SYDNEY GRAMMAR SCHOOL

NAME

MATHEMATICS MASTER

2021 **Trial Examination**

Form VI Mathematics Extension 1

Wednesday 18th August 2021
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 70

Section I (10 marks) Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.
- Write your name and master on each page.

Section II (60 marks) Questions 11 – 14
- Because of the nature of this task, greater weight than normal will be placed on working. Clear reasoning and full calculations are required.
- Record your answers on the writing paper provided.
- Start each question on a new page.
- Write your name and master on each page.

Your sheets must be ORDERED
then scanned and uploaded in a
SINGLE PDF FILE
to the Schoology page of your Mathematics class

Checklist
- Reference sheet
- Writing paper and multiple-choice answer sheet.

Writer: NJL

Section I

Questions in this section are multiple-choice.
Choose the single best answer for each question and record it on the provided answer sheet.

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1. Given the points $A(-2,-5)$ and $B(0,7)$, which of the following represents the position vector $\overrightarrow{BA}$?

   (A) $-2\underset{\sim}{i} - 12\underset{\sim}{j}$

   (B) $2\underset{\sim}{i} + 12\underset{\sim}{j}$

   (C) $-2\underset{\sim}{i} + 2\underset{\sim}{j}$

   (D) $2\underset{\sim}{i} + 2\underset{\sim}{j}$

2. Which of the following is the coefficient of the x^2 term in the expansion of $(2x+5)^6$?

   (A) 15

   (B) 6000

   (C) 37500

   (D) 2500

3. Given that $N = 60 + 50e^{kt}$, which of the following is equal to $\dfrac{dN}{dt}$?

   (A) $10t(6 + 5ke^{kt})$

   (B) $k(N - 60)$

   (C) $k(60 - N)$

   (D) $60t + \dfrac{50e^{kt}}{k}$

4. The logistic equation
$$\frac{dP}{dt} = a\left(1 - \frac{P}{k}\right)P$$
can be used to model the size of a population P, over time t. If a and k are both positive constants and $0 < P < k$, which of the following statements must be true?

   (A) the population is increasing

   (B) the population is constant

   (C) the population is decreasing

   (D) none of the above

5. Let $P(x) = (x-a)Q(x) + r$ for some polynomial $Q(x)$ and some constant r. It is also known that $x = a$ is a double root of $P(x)$.

   Which of the following statements is NOT true?

   (A) $P(a) = 0$

   (B) $r = 0$

   (C) $\deg Q(x) = \deg P(x) - 1$

   (D) $Q(a) \neq 0$

6. Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel. Given that $|\underset{\sim}{a}| = 100$ and $\underset{\sim}{b} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$, which of the following could represent $\underset{\sim}{a}$?

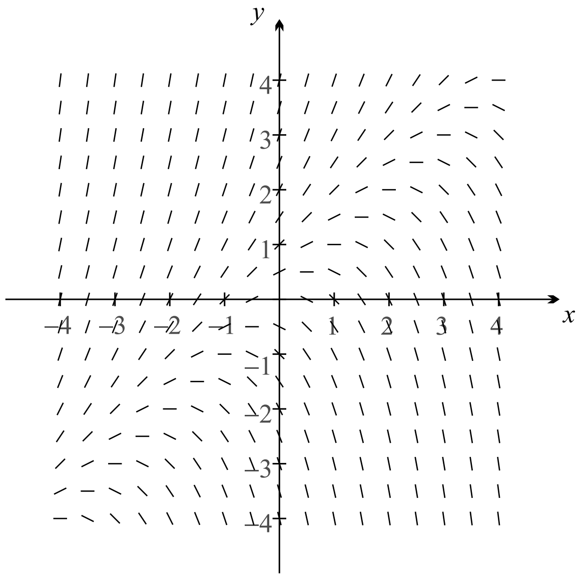
   (A) $96\underset{\sim}{i} + 28\underset{\sim}{j}$

   (B) $14\underset{\sim}{i} + 48\underset{\sim}{j}$

   (C) $700\underset{\sim}{i} + 2400\underset{\sim}{j}$

   (D) $28\underset{\sim}{i} + 96\underset{\sim}{j}$

7.



   Which of the following differential equations could produce the slope field shown above?

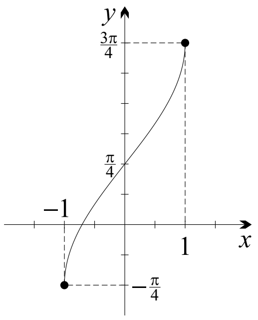
   (A) $\dfrac{dy}{dx} = y^2 - 1$

   (B) $\dfrac{dy}{dx} = x + y$

   (C) $\dfrac{dy}{dx} = y - x$

   (D) $\dfrac{dy}{dx} = y - 1$

8.



Which of the following equations is represented by the graph shown above?

(A) $y = -\cos^{-1} x - \dfrac{\pi}{4}$

(B) $y = \sin^{-1} x + \dfrac{\pi}{4}$

(C) $y = \tan x + \dfrac{\pi}{4}$

(D) $y = \cos^{-1} x - \dfrac{\pi}{4}$

9. Two teams of 3 players each and an umpire are to be formed from seven people. If two of the people cannot be on the same team, which of the following is the number of ways the teams can be formed?

(A) 100

(B) 140

(C) 50

(D) 70

10. Which one of the following is equivalent to $\cos(\tan^{-1} x - \cos^{-1} y)$, given that $x, y \geq 0$?

(A) $\dfrac{x + y\sqrt{1-x^2}}{\sqrt{1+y^2}}$

(B) $\dfrac{y + x(1-y^2)}{1+x^2}$

(C) $\dfrac{x + y(1-x^2)}{1+y^2}$

(D) $\dfrac{y + x\sqrt{1-y^2}}{\sqrt{1+x^2}}$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Record your answers on the writing paper provided.

QUESTION ELEVEN (15 marks) Start a new page. **Marks**

(a) Given $\underset{\sim}{a} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, find:

(i) $|\underset{\sim}{a}|$ $\boxed{1}$

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ $\boxed{1}$

(iii) the exact angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ $\boxed{1}$

(iv) $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$ $\boxed{2}$

(b) Given $y = x \tan^{-1}(2x)$, find y'. $\boxed{2}$

(c) In this question take upwards as positive, $g \doteqdot 10\,\text{ms}^{-2}$ and ignore air resistance.

A tennis ball is hit from a point 2 metres above a level court. The vertical component of the velocity of the ball immediately after being hit is $12\,\text{ms}^{-1}$ upwards.

(i) Write an expression for $\dot{y}$, the vertical component of the ball's velocity t seconds after being hit. $\boxed{1}$

(ii) Hence find how long it takes for the ball to reach its maximum height above the court. $\boxed{1}$

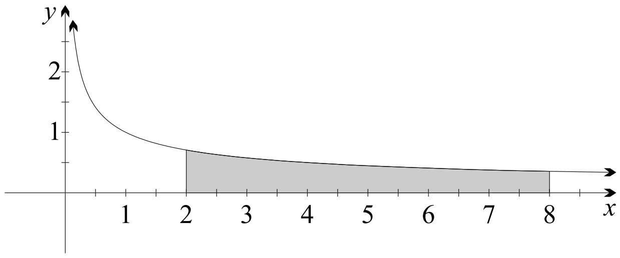
(iii) Write an expression for y, the vertical displacement of the ball above the court t seconds after being hit. $\boxed{1}$

(iv) Find how much time passes between the ball being hit and the ball falling to the court. Give your answer as an exact value in seconds. $\boxed{2}$

(v) The ball initially leaves the racket at an angle of 60° to the horizontal. Calculate the horizontal distance travelled by the ball during its flight. Give your answer in metres, correct to the nearest centimetre. $\boxed{3}$

QUESTION TWELVE (15 marks) Start a new page. **Marks**

(a) 3



The diagram above shows the region enclosed by the graph of $y = \dfrac{1}{\sqrt{x}}$, the vertical lines $x = 2$ and $x = 8$, and the x-axis. This region is rotated about the x-axis.

Find the exact volume of the solid created. Express your answer in simplified form.

(b) Solve the differential equation $\dfrac{dy}{dx} = 3x^2e^{-y}$, given $y(0) = 1$. Express your answer with y as the subject. 3

(c) Use mathematical induction to prove that $6^n - 1$ is divisible by 5 for all positive integers n. 3

(d) Find $\displaystyle\int \frac{1}{9 + 16x^2}\,dx$. 2

(e) (i) Express $3\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. 2

(ii) Hence, or otherwise, solve $3\cos x + 2\sin x = \sqrt{13}$, for $0^\circ \leq x \leq 360^\circ$. Give your answer in degrees correct to two decimal places. 2

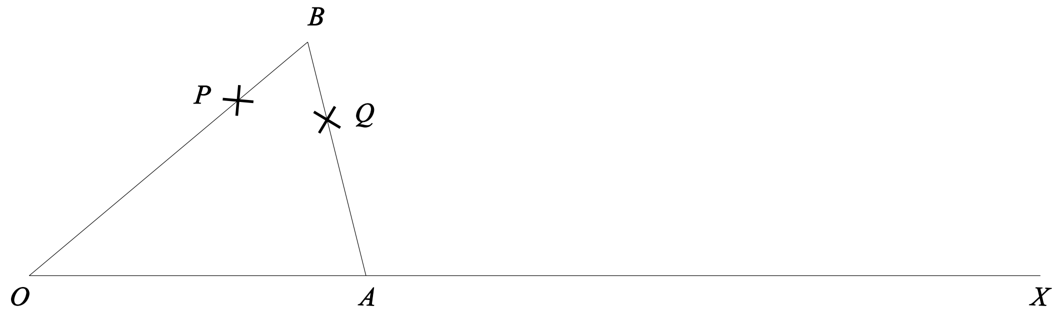
QUESTION THIRTEEN (15 marks) Start a new page. **Marks**

(a) Solve the equation $\boxed{3}$

$$\sin 2x \cos \frac{\pi}{3} + \cos 2x \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \text{for } 0 \leq x \leq 2\pi.$$

(b) Find $\dfrac{d}{dx} \cos^{-1}\left(\dfrac{1}{x} - 1\right)$, expressing your answer in the form $\dfrac{1}{p\sqrt{q-1}}$ where p and q are expressions in terms of x. You may assume $x > \frac{1}{2}$. $\boxed{3}$

(c)



The diagram above shows $\triangle OAB$ and line OAX, where $\overrightarrow{OA} = 3\underset{\sim}{a}$ and $\overrightarrow{OB} = 12\underset{\sim}{b}$.

(i) Express $\overrightarrow{BA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. $\boxed{1}$

(ii) P and Q lie on the lines OB and AB respectively. Given that $OP : PB = 3 : 1$, $AQ : QB = 2 : 1$ and $AX : OA = 2 : 1$, show that P, Q, and X are collinear. (That is, show PQX is a straight line.) $\boxed{3}$

(d) (i) Show that $\sin^3 x = \sin x - \cos^2 x \sin x$. $\boxed{1}$

(ii) Use the substitution $u = \cos x$ to find $\displaystyle\int \cos^2 x \sin x \, dx$. $\boxed{1}$

(iii) Hence find the general solution to the differential equation $\boxed{3}$

$$\operatorname{cosec}^3 x \frac{dy}{dx} = \cos^2 y.$$

You may assume $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

QUESTION FOURTEEN (15 marks) Start a new page. **Marks**

(a) Use double angle formulae to find $\displaystyle\int \sin^2 x \cos^2 x \, dx$. $\boxed{2}$

(b) (i) Let $y = xu$. Given that u is a function of x, find an expression for $\dfrac{dy}{dx}$. $\boxed{1}$

(ii) Hence solve the initial value problem $\dfrac{dy}{dx} = \dfrac{y^2}{2x^2} + \dfrac{1}{2}$, given $y(1) = 2$. $\boxed{3}$

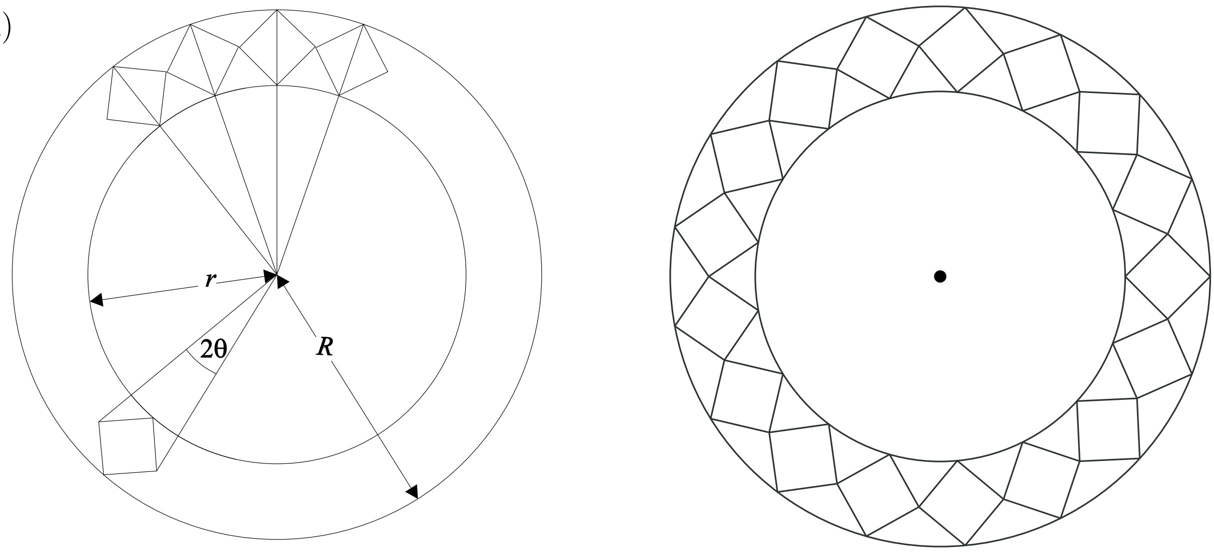
Express your answer with y as the subject.

(c) (i) Given $P(-1) = 0$, factorise $P(x) = 8x^3 + 30x^2 + 37x + 15$. $\boxed{1}$

(ii) Prove by the method of mathematical induction that for all $n \geq 1$, the sum of the squares of the first $2n$ positive integers is given by $\boxed{3}$

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}.$$

(d)



The left diagram above shows two concentric circles with radii R and r, $R > r > 0$. In the annulus between these circles n unit squares are arranged. One diagonal of each square lies on a radius of the larger circle, and the two circles intersect each square at the two vertices at the ends of this diagonal. Each square subtends an angle of 2θ at the common centre of the circles. Each adjacent pair of squares intersect precisely once at a common vertex. The case $n = 17$ is shown completed to the right.

(i) Show that $r = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\tan\theta} - 1\right)$. $\boxed{1}$

(ii) Find a similar expression for R and hence show that $\boxed{2}$

$$\frac{R}{r} = \frac{1 + \tan\theta}{1 - \tan\theta}.$$

(iii) Show that this ratio is an increasing function of θ and hence find its maximum value over all possible values of n. $\boxed{2}$

END OF PAPER

Form VI Ex1 Trial 2021 Sltns

1.) $A(-2,-5)$ $B(0,7)$

$\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$

$= (-2\underset{\sim}{i} - 5\underset{\sim}{j}) - (0\underset{\sim}{i} + 7\underset{\sim}{j})$

$= -2\underset{\sim}{i} - 12\underset{\sim}{j}$

$\therefore$ (A)

2.) $(2x+5)^6 = {}^6C_0(2x)^6(5)^0$

$+ {}^6C_1(2x)^5(5)^1$

$+ {}^6C_2(2x)^4(5)^2$

$+ {}^6C_3(2x)^3(5)^3$

$\Longrightarrow + {}^6C_4(2x)^2(5^4)$

$+ \dots$

${}^6C_4 \times 2^2 \times 5^4 = 37500$

(C)

3.) $N = 60 + 50e^{kt}$

$\frac{dN}{dt} = 50ke^{kt}$

$= k \times 50e^{kt}$

$= k(N-60)$

$\therefore$ (B)

4.) If $P-K<0$

$P<K$

$0<\frac{P}{K}<1$

$1-\frac{P}{K}>0$

$\therefore \frac{dP}{dt}>0$

$\therefore$ population is increasing

$\therefore$ (A)

5) If $(x-a)^2$ is a factor and r is a constant, then $(x-a)$ must be a factor of $Q(x)$.

$\therefore \quad Q(a)=0$.

(D)

6) $|\underset{\sim}{a}| = 100 \qquad \underset{\sim}{b} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$

$|\underset{\sim}{b}| = \sqrt{7^2 + 24^2}$

$= 25$

$4|\underset{\sim}{b}| = |\underset{\sim}{a}|$

$\underset{\sim}{a} = 4 \times 7\underset{\sim}{i} + 4 \times 24\underset{\sim}{j}$

$\underset{\sim}{a} = 28\underset{\sim}{i} + 96\underset{\sim}{j}$

(D)

7) • Consider $\frac{dy}{dx} = y^2 - 1$. at $y=0$ $\frac{dy}{dx} = 1$

but on slope field $\frac{dy}{dx} \neq C$ at $y=0$.

$\therefore$ Not (A).

• Consider $\frac{dy}{dx} = x + y$ at $(-4,-4)$.

$\frac{dy}{dx} \neq -8$ on slope field $\therefore$ Not (B).

• Consider $\frac{dy}{dx} = y - x$ at $(-4,-4)$.

$\frac{dy}{dx} = 0$, agrees with slope field.

at $y=0$ $\frac{dy}{dx} = -x$. Agrees with slope field.

at $x=0$ $\frac{dy}{dx} = y$. Also agrees.

• Check $\frac{dy}{dx} = y - 1$ at $(-4,-4)$.

$\frac{dy}{dx} \neq -5$ $\therefore$ Not (D)

$\therefore$ (C)

8.) This is a translation of $y = \sin^{-1} x$ by $\pi/4$ upwards

$\therefore \quad y = \sin^{-1} x + \pi/4$

$\therefore$ (B)

9.) Let A and B be the players in separate teams.

2 cases: A and B are both in separate teams OR one of them is an umpire and the other is in a team.

Case 1: Both in separate teams.

$$\underline{A}\ \underline{\ \ }\ \underline{\ \ } \qquad \underline{B}\ \underline{\ \ }\ \underline{\ \ } \qquad \underline{\ \ }$$

$$^5C_2 \quad \times \quad {}^3C_2 \quad \times \quad 1$$

Case 2: Either A or B is an umpire. Once A or B is chosen there are 6 players left to form 2 teams.

$^2C_1 \times {}^6C_3 \times {}^3C_3$

But forming a team of 3 automatically forms the other team so we have to divide by 2.

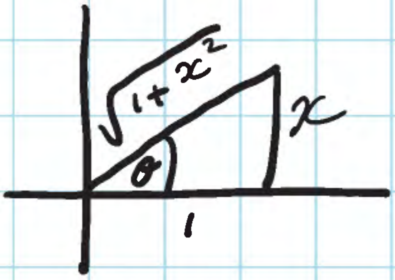
$$\therefore \quad {}^5C_2 \times {}^3C_2 + \frac{{}^2C_1 \times {}^6C_3 \times {}^3C_3}{2}$$

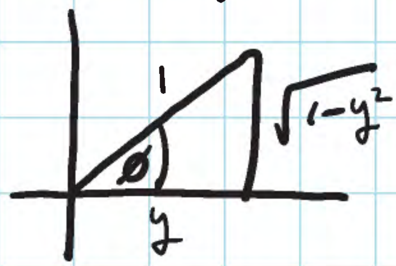
$$= \underline{\underline{50}}$$

10) $\cos\left[\tan^{-1}x - \cos^{-1}y\right]$

let $\theta = \tan^{-1}x$ $\qquad\qquad$ $\phi = \cos^{-1}y$

$\tan\theta = x$ $\qquad\qquad$ $\cos\phi = y$





$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$ $\qquad$ $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

$\cos\phi = y$ $\qquad$ $\sin\phi = \sqrt{1-y}$

$$\cos(\theta-\phi) = \frac{1}{\sqrt{1+x^2}} \times y + \frac{x}{\sqrt{1+x^2}} \times \sqrt{1-y^2}$$

$$= \frac{y + x\sqrt{1-y^2}}{\sqrt{1+x^2}}$$

(D)

11.) a) $\underset{\sim}{a} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

i) $|\underset{\sim}{a}| = \sqrt{(-4)^2 + 1^2}$
$= \sqrt{16+1}$
$= \sqrt{17}$ ✓

ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$= (-4) \times (-3) + (1) \times 5$
$= 12 + 5$
$= 17$ ✓

iii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$
$17 = \sqrt{17} \times \sqrt{(-3)^2 + 5^2} \times \cos\theta$
$\cos\theta = \frac{17}{\sqrt{17}\sqrt{34}}$

$\cos\theta = \frac{\sqrt{17}}{\sqrt{34}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \pi/4$ (or 45°) ✓

iv) $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}$

$$= \frac{17}{(-3)\times(-3)+5\times5} \times \begin{bmatrix} -3 \\ 5 \end{bmatrix} \checkmark$$

$$= \frac{17}{34}\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 \\ 5/2 \end{bmatrix} \checkmark$$

b) $u = x \qquad v = \tan^{-1}(2x)$

$u' = 1 \qquad v' = \dfrac{2}{1+4x^2} \checkmark$

$y' = vu' + uv'$

$= \tan^{-1}(2x) + \dfrac{2x}{1+4x^2} \checkmark$

c) i) $\ddot{y} = -10$

$\dot{y} = -10t + c$

$t = 0,\ \dot{y} = 12,\ c = 12$

$\dot{y} = -10t + 12$ ✓

ii) at max height $\dot{y} = 0$

$0 = -10t + 12$

$\frac{12}{10} = t$

$t = 1.2$ secs ✓

iii) $y = \frac{-10t^2}{2} + 12t + c$

$t = 0,\ y = 2,\ c = 2$

$y = -5t^2 + 12t + 2$ ✓

iv) $0 = -5t^2 + 12t + 2$

$0 = 5t^2 - 12t - 2$

$$t = \frac{12 \pm \sqrt{12^2 - 4 \times 5 \times (-2)}}{2 \times 5}$$ ✓

$$t = \frac{12 \pm \sqrt{184}}{10}$$

$$t = \frac{12 \pm 2\sqrt{46}}{10}$$

$t > 0$

$\therefore t = \frac{6 + \sqrt{46}}{5}$ secs. ✓

(or equivalent method).

v) $V \sin 60 = 12$

$$\frac{\sqrt{3}}{2} V = 12$$

$$V = \frac{12 \times 2\sqrt{3}}{3}$$

$$V = 8\sqrt{3} \text{ ms}^{-1}$$ ✓

$$\dot{x} = 8\sqrt{3} \cos 60$$

$$= 8\sqrt{3} \times \tfrac{1}{2}$$

$$= 4\sqrt{3} \text{ ms}^{-1}$$ ✓

for $t = \dfrac{6+\sqrt{46}}{5}$

$$x = 4\sqrt{3} \times \tfrac{1}{5} \times (6+\sqrt{46})$$

$$= \frac{4}{5}\sqrt{3}(6+\sqrt{46})$$

$$= 17.71 \text{ m}$$ ✓

12) a) $y = \frac{1}{\sqrt{x}}$,

$y^2 = \frac{1}{x}$

$\int_2^8 \pi y^2 \, dx$

$= \pi \int_2^8 \frac{1}{x} \, dx$ ✓

$= \pi \left[\ln|x| \right]_2^8$ ✓

$= \pi \left[\ln(8) - \ln(2) \right]$

$= \pi \ln 4$ ✓

b) $\frac{dy}{dx} = 3x^2 e^{-y}$ $\quad y(0) = 1$

$e^y dy = 3x^2 dx$

$\int e^y dy = \int 3x^2 dx$ ✓

$e^y = x^3 + c$

$y = \ln(x^3 + c)$ ✓

$y(0) = 1$

$1 = \ln(c)$

$c = e$

$y = \ln(x^3 + e)$ ✓

c) A. When $n=1$, $6^1-1=5$, which is divisible by 5 so the statement is true for $n=1$

B. Suppose $k \geqslant 1$ is an integer for which the statement is true. That is, suppose

$$6^k - 1 = 5m$$

We prove the statement for $n=k+1$. That is, we prove $6^{k+1}-1$ is divisible by 5.

$$\begin{aligned} & 6^{k+1} - 1 \\ &= 6^k \times 6 - 1 \\ &= (5m+1)6 - 1 \quad ✓ \\ &= 6m \times 5 + 6 - 1 \quad \text{, by the induction hypothesis.} \\ &= 6m \times 5 + 5 \\ &= 5(6m+1) \quad ✓ \end{aligned}$$

which is divisible by 5.

C. It follows from parts A and B by mathematical induction that the statement is true for all whole numbers n.

✓ for structure of argument A, B and C.

d) $\int \frac{1}{9+16x^2}\,dx$

from ref. sheet: $\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + C$

$\int \frac{1}{3^2+[4x]^2}\,dx$

$= \frac{1}{4}\int \frac{4}{3^2+[4x]^2}\,dx$ ✓

$= \frac{1}{4} \times \frac{1}{3}\tan^{-1}\left(\frac{4x}{3}\right)+C$

$= \frac{1}{12}\tan^{-1}\left(\frac{4x}{3}\right)+C$ ✓

e) i) $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$3\cos x + 2\sin x = R\cos\alpha\cos x + R\sin\alpha\sin x$

$3 = R\cos\alpha \qquad 2 = R\sin\alpha$

$3^2+2^2 = R^2(\cos^2\alpha+\sin^2\alpha)$

$13 = R^2$

$R = \sqrt{13}$. ✓

$2 = \sqrt{13}\sin\alpha \qquad 0<\alpha<90°$

$\alpha = \sin^{-1}(2/\sqrt{13}) \approx 33.69°$ (to 2 d.p)

$\therefore\ 3\cos x + 2\sin x = \sqrt{13}\cos\left(x - \sin^{-1}\left(2/\sqrt{13}\right)\right)$

$3\cos x + 2\sin x = \sqrt{13}\cos(x - 33.69°)$ ✓

ii) If $3\cos x + 2\sin x = \sqrt{13}$ $\quad 0° \leq x \leq 360°$

$\sqrt{13}\cos(x - 33.69) = \sqrt{13}$ ✓

for $-33.69 \leq x - 33.69 \leq 326.31$

$\cos\theta = 1$ for $-33.69 \leq \theta \leq 326.31$

$\theta = 0$

$x = 33.69°$ ✓

(Don't penalise rounding)

13.) a) LHS: $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin\left(2x + \frac{\pi}{3}\right) = \sqrt{3}/2$ ✓

$0 \leq x \leq 2\pi$

$\frac{\pi}{3} \leq 2x + \pi/3 \leq 4\pi + \pi/3$

$\frac{\pi}{3} \leq \theta \leq 13\pi/3$

$\sin\theta = \sqrt{3}/2$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \left(2\pi + \frac{\pi}{3}\right), \left(3\pi - \pi/3\right), \left(4\pi + \frac{\pi}{3}\right)$ ✓

$2x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \left(2\pi + \frac{\pi}{3}\right), \left(3\pi - \pi/3\right), \left(4\pi + \frac{\pi}{3}\right)$

$2x = 0, \frac{\pi}{3}, 2\pi, \frac{7\pi}{3}, 4\pi$

$x = 0, \pi/6, \pi, \frac{7\pi}{6}, 2\pi$ ✓

(Award 2 marks for

$x = 0, \pi, 2\pi$

or 2 marks for $x = \pi/6, 7\pi/6$ and 2π)

Alternatively:

$$\sin 2x \cos \tfrac{\pi}{3} + \cos 2x \sin \tfrac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad 0 \le x \le 2\pi$$

$$2\sin x \cos x \times \tfrac{1}{2} + (\cos^2 x - \sin^2 x) \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$ ✓

$$\sin x \cos x + \frac{\sqrt{3}}{2}(\cos^2 x - \sin^2 x) = \frac{\sqrt{3}}{2}$$

$$\frac{\sin x \cos x}{\cos^2 x} + \frac{\sqrt{3}}{2}\left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x}\right) = \frac{\sqrt{3}}{2\cos^2 x}$$

$$\tan x + \frac{\sqrt{3}}{2}(1 - \tan^2 x) = \frac{\sqrt{3}}{2}\sec^2 x$$

$$\tan x + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\tan^2 x - \frac{\sqrt{3}}{2}\sec^2 x = 0$$

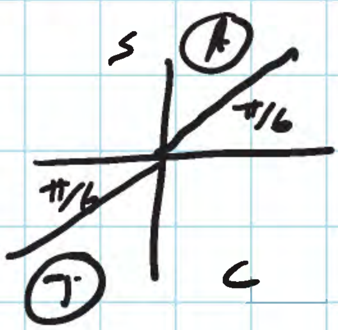
$$2\tan x + \sqrt{3} - \sqrt{3}\tan^2 x - \sqrt{3}\sec^2 x = 0$$

$$2\tan x + \sqrt{3} - \sqrt{3}\tan^2 x - \sqrt{3}(1 + \tan^2 x) = 0$$

$$2\tan x + \sqrt{3} - \sqrt{3} - \sqrt{3}\tan^2 x - \sqrt{3}\tan^2 x = 0$$ ✓

$$-2\sqrt{3}\tan^2 x + 2\tan x = 0$$

$$-2\tan x(\sqrt{3}\tan x - 1) = 0$$



$\tan x = 0$ $\qquad$ $\tan x = 1/\sqrt{3}$

$x = 0, \pi, 2\pi$ $\qquad$ $x_r = \pi/6$

$\qquad\qquad x = \frac{\pi}{6}, \frac{7\pi}{6}$

$$\therefore x = 0, \pi/6, \pi, 7\pi/6, 2\pi$$ ✓

b) $\frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}-1\right)\right) = \frac{1}{p\sqrt{q-1}}$

if $y = \cos^{-1} f(x)$, $\frac{dy}{dx} = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

$$y = \cos^{-1}\left(\frac{1}{x}-1\right)$$
$$= \cos^{-1}\left(x^{-1}-1\right)$$

$$\frac{dy}{dx} = \frac{-(-x^{-2})}{\sqrt{1-\left[\frac{1}{x}-1\right]^2}}$$ ✓

$$= \frac{\frac{1}{x^2}}{\sqrt{1-\left[\frac{1-x}{x}\right]^2}}$$

$$= \frac{\frac{1}{x^2}}{\sqrt{1-\left[\frac{1-x}{x}\right]^2}}$$

$$= \frac{\frac{1}{x^2}}{\sqrt{\frac{x^2}{x^2}-\frac{(1-x)^2}{x^2}}}$$ (or any valid simplification). ✓

$$= \frac{\frac{1}{x^2}}{\sqrt{\frac{1}{x^2}\left(x^2-(1-x)^2\right)}}$$

$$= \frac{\frac{1}{x^2}}{\frac{1}{x}\sqrt{x^2-(1-2x+x^2)}} = \frac{1}{x\sqrt{2x-1}}$$ ✓

$\therefore\ p = x \quad q = 2x$

c) i) $\overrightarrow{BA} = -12\underset{\sim}{b} + 3\underset{\sim}{a}$ ✓

ii) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OA} + \overrightarrow{AQ}$

$= -\frac{3}{4}\overrightarrow{OB} + \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= -\frac{3}{4}(12\underset{\sim}{b}) + 3\underset{\sim}{a} - \frac{2}{3}(-12\underset{\sim}{b} + 3\underset{\sim}{a})$

$= -9\underset{\sim}{b} + 3\underset{\sim}{a} + 8\underset{\sim}{b} - 2\underset{\sim}{a}$

$= -\underset{\sim}{b} + \underset{\sim}{a}$

$= \underset{\sim}{a} - \underset{\sim}{b}$ ✓

$\overrightarrow{QX} = \overrightarrow{QA} + \overrightarrow{AX}$

$= \frac{2}{3}(-12\underset{\sim}{b} + 3\underset{\sim}{a}) + 6\underset{\sim}{a}$

$= -8\underset{\sim}{b} + 2\underset{\sim}{a} + 6\underset{\sim}{a}$

$= 8\underset{\sim}{a} - 8\underset{\sim}{b}$ ✓

$\therefore 8\overrightarrow{PQ} = \overrightarrow{QX}$ ✓ (Award mark if $\overrightarrow{PX} = 9\overrightarrow{PQ}$)

$\therefore \overrightarrow{PQ} \parallel \overrightarrow{QX}$

Q is common

$\therefore$ PQX is a straight line.

d) i) $LHS = \sin^3 x$

$= \sin x(1-\cos^2 x)$ ✓

$= \sin x - \cos^2 x \sin x$

ii) $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\int \cos^2 x \sin x \, dx = -\int u^2 \frac{du}{dx} dx$

$= -\frac{u^3}{3} + c$

$= -\frac{1}{3}\cos^3 x + C$ ✓

$$\boxed{\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C}$$

iii) $\operatorname{cosec}^3 x \frac{dy}{dx} = \cos^2 y$

$\int \frac{1}{\cos^2 y} dy = \int \frac{1}{\operatorname{cosec}^3 x} dx$ ✓

$\int \sec^2 y \, dy = \int \sin^3 x \, dx$

$\tan y + A = \int \sin^2 x \cdot \sin x \, dx$ (for $\tan y$) ✓

$\tan y + A = \int (1-\cos^2 x)\sin x \, dx$

$\tan y + A = \int \sin x - \cos^2 x \sin x \, dx$

$\tan y + A = -\cos x - \left[-\frac{\cos^3 x}{3}\right] + B$

$\tan y = -\cos x + \frac{\cos^3 x}{3} + c$ ✓

$y = \tan^{-1}\left(-\cos x + \frac{\cos^3 x}{3} + c\right)$

(Don't penalise lack of constant).

14) a) $\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$

$= \frac{1}{4}\int 1 - \cos^2(2x) \, dx$

$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos(4x) \, dx$ ✓

$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin(4x)\right) + C$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$ ✓

b) i) $y = xu$ if $y = uv$ $y' = vu' + uv'$

$\frac{dy}{dx} = u + \frac{x\,du}{dx}$ ✓

ii) $y' = \frac{y^2}{2x^2} + \frac{1}{2}$ $y(1) = 2$

Since $y' = u + xu'$ and $y = xu$

$u + xu' = \frac{(xu)^2}{2x^2} + \frac{1}{2}$ ✓

$u + xu' = \frac{u^2}{2} + \frac{1}{2}$

$xu' = \frac{u^2}{2} + \frac{1}{2} - u$

$$xu' = \frac{u^2 - 2u + 1}{2}$$

$$xu' = \frac{1}{2}(u-1)^2$$

$$x\frac{du}{dx} = \frac{1}{2}(u-1)^2$$

$$\frac{1}{(u-1)^2}\frac{du}{dx} = \frac{1}{2x}$$

$$\int \frac{1}{(u-1)^2}\,du = \int \frac{1}{2x}\,dx$$

$$\int (u-1)^{-2}\,du = \int \frac{1}{2x}\,dx \quad \checkmark$$

$$\frac{(u-1)^{-1}}{-1(1)} = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$-\frac{1}{u-1} = \frac{1}{2}\ln|x| + c$$

$$\frac{1}{1-u} = \frac{1}{2}\ln|x| + c$$

$$\frac{1}{1-y/x} = \frac{1}{2}\ln|x| + c$$

$$1 - y/x = \frac{1}{\frac{1}{2}\ln|x| + c}$$

$$1 - \frac{1}{\frac{1}{2}\ln|x| + c} = \frac{y}{x}$$

$$y = x - \frac{x}{\frac{1}{2}\ln|x| + C}$$

$$y = x - \frac{x}{\ln\sqrt{|x|} + C}$$ ✓

c) i) Given $P(-1) = 0$, $(x+1)$ is a factor

$$\begin{array}{r l}
 & 8x^2 + 22x + 15 \\
x + 1 \, \big) & 8x^3 + 30x^2 + 37x + 15 \\
 & \underline{8x^3 + 8x^2} \\
 & 22x^2 + 37x \\
 & \underline{22x^2 + 22x} \\
 & 15x + 15 \\
 & \underline{15x + 15} \\
 & 0
\end{array}$$

$\therefore P(x) = (x+1)(8x^2 + 22x + 15)$

$P(x) = (x+1)(4x+5)(2x+3)$ ✓

c) ii) A. Let $n=1$ $RHS = \frac{1(2+1)(4+1)}{3}$

$= \frac{3 \times 5}{3}$

$= 5$

for $n=1$ $2n = 2$

Sum of squares of 1st 2 positive integers:

$LHS = 1^2 + 2^2$

$= 5$

$LHS = RHS$ for $n=1$

So the statement is true for $n=1$.

B. Suppose $k \geq 1$ is a positive integer for which the statement is true. That is

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$$

We prove the statement for $n = k+1$.

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}$$

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{1}{3}(k+1)(4k+5)(2k+3)$$

$$LHS = 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2(k+\tfrac{1}{2}))^2 + (2(k+1))^2 \checkmark$$

$$= 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2$$

$$= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3}$$ (By the induction hypothesis) $\checkmark$

$$= \frac{1}{3}\left(8k^3 + 6k^2 + k + 12k^2 + 12k + 3 + 12k^2 + 24k + 12\right)$$

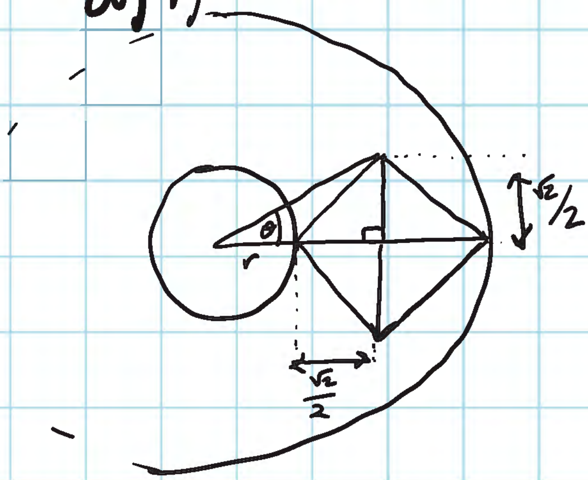
$$= \frac{1}{3}\left(8k^3 + 30k^2 + 37k + 15\right)$$

$$= \frac{1}{3}(k+1)(4k+5)(2k+3) \checkmark$$

$=$ RHS as required.

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers.

d) i)



$$\tan\theta = \frac{\sqrt{2}}{2} \div \left(r + \frac{\sqrt{2}}{2}\right)$$

$$\tan\theta = \frac{\sqrt{2}}{2} \div \left(\frac{2r+\sqrt{2}}{2}\right)$$

$$\tan\theta = \frac{\sqrt{2}}{2} \times \frac{2}{2r+\sqrt{2}}$$

$$\tan\theta = \frac{\sqrt{2}}{2r+\sqrt{2}}$$

$$2r\tan\theta + \sqrt{2}\tan\theta = \sqrt{2}$$

$$2r\tan\theta = \sqrt{2} - \sqrt{2}\tan\theta$$

$$2r = \frac{\sqrt{2}}{\tan\theta} - \sqrt{2}$$

$$r = \frac{\sqrt{2}}{2\tan\theta} - \frac{\sqrt{2}}{2}$$

$$r = \frac{\sqrt{2}}{2}\left(\frac{1}{\tan\theta} - 1\right)$$

$$r = \frac{2}{2\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right)$$

$$r = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right)$$

✓

ii) $R = r + \sqrt{2}$

$$R = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right) + \sqrt{2}$$

$$R = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right) + \frac{2}{\sqrt{2}}$$

$$R = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1 + 2\right)$$

$$R = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} + 1\right) \quad \checkmark$$

$$\therefore \quad \frac{R}{r} = \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} + 1\right)}{\frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right)}$$

$$= \frac{\frac{1}{\tan\theta} + \frac{\tan\theta}{\tan\theta}}{\frac{1}{\tan\theta} - \frac{\tan\theta}{\tan\theta}}$$

$$= \frac{\frac{1+\tan\theta}{\tan\theta}}{\frac{1-\tan\theta}{\tan\theta}} \quad \checkmark$$

$$= \frac{1+\tan\theta}{1-\tan\theta}$$

iii) let $y = \frac{R}{r}$

$$y = \frac{1+\tan\theta}{1-\tan\theta}$$

Let $u = 1+\tan\theta$ $\quad v = 1-\tan\theta$

$u' = \sec^2\theta$ $\quad v' = -\sec^2\theta$

$$y' = \frac{vu' - uv'}{v^2}$$

$$y' = \frac{\sec^2\theta(1-\tan\theta) - -\sec^2\theta(1+\tan\theta)}{(1-\tan\theta)^2}$$

$$y' = \frac{\sec^2\theta - \sec^2\theta\tan\theta + \sec^2\theta + \sec^2\theta\tan\theta}{(1-\tan\theta)^2}$$

$$y' = \frac{2\sec^2\theta}{(1-\tan\theta)^2}$$

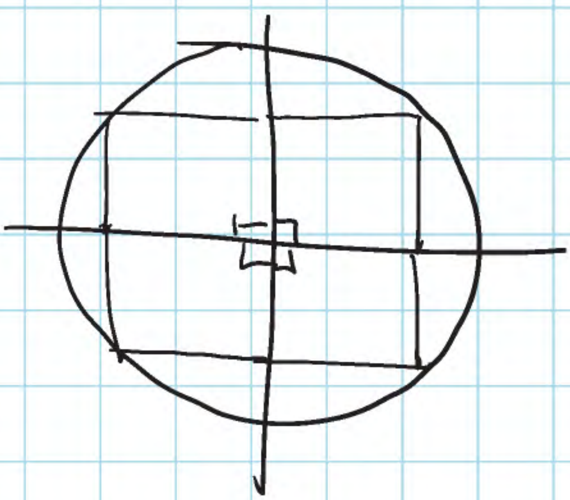
$(\sec\theta)^2 > 0$

$(1-\tan\theta)^2 > 0$ for $\theta \neq \pi/4$

$\therefore y' > 0$ for $\theta \neq \pi/4$

$\therefore y = \frac{R}{r}$ is an increasing function. ✓

$\therefore$ y_{max} when θ is a maximum.
If $\theta = \pi/4$, $2\theta = \pi/2$ and angle at centre is a right angle.



$\therefore$ Only 4 squares possible but $r=0$.

$\therefore$ $\underline{n \geq 5}$.

For $n=5$ $\theta = \pi/5$

$$y_{max} = \frac{1+\tan(\pi/5)}{1-\tan(\pi/5)}$$

$y_{max} = 6.31$ (to 2 d.p) ✓